



UNIVERSITY OF  
**LATVIA**

Summary of  
Doctoral Thesis

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**Insa Cremer**

**ORDERINGS AND  
STRUCTURE OF STRONG  
RICKART RINGS**

Riga 2025



UNIVERSITY OF  
**LATVIA**

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**FACULTY OF SCIENCE  
AND TECHNOLOGY**

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STRONG RICKART RINGS**

SUMMARY OF DOCTORAL THESIS

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The thesis is available at the Library of the University of Latvia, Kalpaka blvd. 4, Riga.

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# Abstract

This thesis is concerned with several partial orders on certain classes of Rickart rings (possibly one-sided ones) satisfying that or other *strongness condition* – a condition which ensures that, without the need for an involution, the ring resembles a Rickart  $*$ -ring in certain aspects. This approach allows for the transfer of knowledge from the field of Rickart  $*$ -rings to a wider class of rings not necessarily having involution, showing that the results obtained for Rickart  $*$ -rings can often be at least partly preserved.

In this spirit, we introduce a version of the strong right star order to so-called *right-strong Rickart rings*, obtaining a relatively orthocomplemented poset like for a Rickart  $*$ -ring.

Parts of the thesis focus on investigating conditions for the existence of meets and joins under several partial orders on Rickart rings satisfying a strongness condition. For example, we obtain a series of equivalent conditions under which two elements of a so-called *strong Rickart ring* have the join under the star order.

By a unified approach that involves weak BCK-algebras, we obtain, for most of the studied partial orders, conditions under which the respective poset is a meet semilattice.

Another focus of the thesis is on strong semilattice decompositions of certain reducts of reduced Rickart rings. Among other things, it is shown that, given a reduced Rickart ring  $\langle R, +, \cdot, 1 \rangle$  with the unary operation  $\circ$  mapping every element to its minimal idempotent duplicator, the algebra  $\langle R, \cdot, \circ, 1 \rangle$  is a strong semilattice of right-cancellative D-semigroups, and that this strong semilattice representation is essentially unique.

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**Keywords:** Rickart ring, Rickart  $*$ -ring, strong Rickart ring, reduced Rickart ring, star order, one-sided star order, diamond order, sharp order, Abian order, join, meet, meet semilattice, relatively orthocomplemented poset, weak difference, weak BCK-algebra, strong semilattice decomposition

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# Introduction

## *Directions of research on partial orders on Rickart rings*

Several partial orders on Rickart rings and Rickart  $*$ -rings have been studied for more than 40 years. They were also often studied on specific examples, like matrix rings or the ring of bounded linear operators on a Hilbert space, or on special cases of Rickart rings which have a simpler structure, like reduced Rickart rings.

Different directions of research include investigating the connections and similarities between the various partial order (see, for example, [Dolinar and Marovt, 2018]) or the links of the partial orders to ring theoretic notions, for example, generalized inverses (see [Marovt, 2015]). Moreover, the general order-theoretic properties of the partial orders are a field of interest, for example, existence of meets and joins, as well as complementations, and the structure of the so-called initial segments (down-sets under some element). Another direction is transferring results from the more specific setting of Rickart  $*$ -rings to the more general Rickart rings or even one-sided Rickart rings (in some cases, even semigroups, see [Čirulis, 2019]).

**Research questions for this thesis:** This thesis is mostly part of the research directions mentioned in the last two sentences of the previous paragraph, but another aim is to prove stronger results which usually hold only in specific Rickart rings.

## *Strong Rickart rings*

*Strong Rickart rings* were introduced in [Čirulis, 2016] as a "star-free" generalization of Rickart  $*$ -rings (i.e., a generalization that does not require an involution). They are equipped with two unary operations called left, respectively right, focal operation. Every Rickart  $*$ -ring, as well as every reduced Rickart ring, can be seen as a strong Rickart ring. A strong Rickart ring is by definition a (both-sided) Rickart ring, but there are also one-sided versions of this notion.

In this thesis, we study star-free versions of several partial orders: The star order and the weak right star order were generalized to a star-free setting in [Čirulis, 2015c] and [Čirulis, 2016], and for the diamond order, this was done

in Cīrulis [2017]. A generalization of the strong right star order to so-called *right-strong Rickart rings* is provided by this thesis.

### ***The aim of the thesis***

The aim of the thesis is to investigate whether certain properties which are known for these partial orders in Rickart  $*$ -rings also hold in the corresponding star-free settings in some way, to adapt certain notions which are used in the study of the star order to the one-sided star orders and to find comparable one-sided versions of results which are known for the star order.

There are three main results: First, it is proved that a right-strong Rickart ring under the strong right star order is a relatively orthocomplemented poset. Second, by a unified approach involving weak BCK-algebras, we find necessary and sufficient conditions for a suitable type of Rickart ring equipped to be a meet semilattice under several of the partial orders treated in the thesis. The third result deals with the special case of reduced Rickart rings. We unite a strong semilattice decomposition of the multiplicative semigroup of a reduced Rickart ring  $R$  (see also [Fountain, 1976]) with the strong semilattice decomposition of  $R$  equipped with the unary operation which maps each element to its minimal idempotent duplicator in the sense of N.V. Subrahmanyam.

### ***Methods***

Methods from different branches of algebra are used. In obtaining results for the strong right star order, we often make use of its close connection to the weak right star order, which has already been studied on strong Rickart rings. The meet semilattice conditions for several partial orders are found by a generalizing approach that uses weak BCK-algebras and similar structures. Moreover, a semigroup theoretic technique is used in order to study the structure of reduced Rickart rings.

### ***Approbation of the results and contribution of the author***

Results obtained as a part of the thesis have been presented in 11 international conferences and six domestic conferences (for details, see the list of attended conferences in the end of the thesis).

The main results of this thesis are contained in four scientific papers: Cīrulis and Cremer [2022], Cremer [2024], Cremer and Marovt [ND] and Cremer [2025] (see also the list of author's publications).

Results from Sections 4.3 and 4.5 in Chapter 4 were published in the first article Cīrulis and Cremer [2022], which was joint work with Professor Jānis Cīrulis. In that publication, the author proved Proposition 3.2, Theorem 5.3 and a part of Theorem 5.4. She also checked the article for mistakes. The second

article Cremer [2024] consists of content from Sections 3.1, 4.2 and 5.1. The results which will be published in the third article Cremer [2025] can be found in Sections 7.1 and 7.1. Finally, the results from Chapter 6 will be published in the fourth article Cremer and Marovt [ND]. In this joint work with Professor Janko Marovt, the author proved all the results and did most of the writing and editing.

## Structure of the thesis

The thesis has 165 pages. It is divided into two parts: Part I deals with the preliminaries and contains the first two chapters (Chapters 1 and 2), while Part II contains the original results (Chapters 3 to 7). There are also three appendices (Appendix A, Appendix B and Appendix C), which some prerequisites that were not included in Part I because they are not directly related to partial orders on Rickart rings.

First, in Chapter 1, we introduce the relevant types of Rickart rings. The second preliminary chapter is Chapter 2, which deals with specific partial orders on different types of Rickart rings, in particular the star order, the one-sided star orders and the weak one-sided star orders.

Part II begins in Chapter 3, where we define the strong right star order on right-strong Rickart ring and prove that it is indeed a partial order. We also introduce another partial order on a right-strong right Rickart ring which is weaker than the strong right star order, but stronger than the weak right star order, and thus can be regarded as a third generalization of the right star order. In Chapter 4, we deal with conditions for the existence of joins and meets under different partial orders on Rickart rings satisfying some strongness condition. In Chapter 5, we investigate the order properties of a right-strong Rickart ring with the strong right star order. The main result is that this poset is relatively orthocomplemented. We also provide order embeddings of initial segments into certain ordered sets which are related to idempotents of the ring and we characterize the images of such isomorphisms. In Chapter 6, we prove meet semilattice conditions for the one-sided star orders on a strong Rickart ring, as well as for the sharp order on an arbitrary unitary ring. In the last chapter, Chapter 7, we focus on the special case of reduced Rickart rings. The main result is a strong semilattice decomposition of the multiplicative semigroup of a reduced Rickart ring equipped with a unary operation that turns it into a D-semigroup. We also derive an analogous strong semilattice decomposition which uses the skew meet instead of the D-semigroup operation.

# Chapter 1

## Rickart rings and strongness conditions

### 1.1. Rickart rings

A right (left) Rickart ring is a ring  $R$  in which for all  $a \in R$  there exists an idempotent  $e \in R$  such that, for all  $x \in R$ ,  $ax = 0$  if and only if  $ex = x$  ( $xa = 0$  if and only if  $xe = x$ ). Obviously, this means that a ring  $R$  is a right Rickart ring if and only if there exists a unary operation  $'$  such that, for every  $a \in R$ ,  $a'$  is an idempotent with the property that, for every  $x \in R$ ,

$$ax = 0 \text{ iff } a'x = x. \quad (1.1)$$

Such an operation is called *right focal operation* in [Cīrulis, 2016], adapting a term from [Foulis, 1960].

Similarly, a ring  $R$  is a left Rickart ring if and only if there exists a unary operation  $\backslash$  such that, for every  $a \in R$ ,  $a\backslash$  is an idempotent with the property that, for every  $x \in R$ ,

$$xa = 0 \text{ iff } xa\backslash = x. \quad (1.2)$$

Such an operation is called *left focal operation* in [Cīrulis, 2016]. By a *focal operation* we mean an operation which is either right focal or left focal.

**Definition 1.1.** A Rickart ring (respectively, right Rickart ring) equipped with some right focal operation is called *right-focal Rickart* (respectively, *right-focal right Rickart*) ring.

**Definition 1.2.** [Cīrulis and Cremer, 2018] A focal operation on a unitary ring  $R$  is said to be *normal* if  $a'' = 1 - a'$  for all  $a \in R$  (where  $a''$  is a shorthand for  $(a')'$ ). A left or right Rickart ring whose focal operation is normal will sometimes also be called *normal* in this thesis.

**Definition 1.3.** [Maeda, 1960] Let  $R$  be a ring and let  $a \in R$ . An idempotent  $e \in R$  is called a *right idempotent of a* if, for all  $x \in R$ ,  $ax = 0$  holds if and only if  $ex = 0$  holds. It is called *left idempotent of a* if, for all  $x \in R$ ,  $xa = 0$  holds

if and only if  $xe = 0$  holds. We denote the set of all right idempotents of an element  $a$  by  $\text{RI}(a)$  and the set of all left idempotents of  $a$  by  $\text{LI}(a)$ .

**Definition 1.4.** A ring  $R$  is called *reduced* if it has no non-zero nilpotent elements, that is, for every  $a \in R$  and for every  $n \in \mathbb{N}$ ,  $a^n = 0$  implies  $a = 0$ .

**Lemma 1.5.** Cīrulis and Cremer [2020] *A ring with unity is a reduced Rickart ring if and only if it admits a unary operation  $^\circ$  such that*

- (a)  $xx^\circ = x = x^\circ x$ ,
- (b)  $(xy)^\circ = x^\circ y^\circ$ ,
- (c)  $0^\circ = 0$ .

*In this case, the focal operation of the ring is given by  $x' := 1 - x^\circ$ .*

## 1.2. Strongness conditions for Rickart rings and one-sided Rickart rings

**Definition 1.6.** (a) Let  $R$  be a right-focal right Rickart ring (left-focal left Rickart ring). An idempotent  $e \in R$  is called *closed* if it is in the range of the right focal operation (left focal operation) (i.e.,  $e = a'$  for some  $a \in R$ ). We will denote the range of the right focal operation (left focal operation) by  $P_r$ .

- (b) Let  $R$  be a left-right-focal Rickart ring with a pair of normal focal operations such that

$$a^v = a^\wedge \text{ and } a^\wedge = a'' . \quad (1.3)$$

An idempotent  $e \in R$  is called *closed* if it is in the range of the focal operations. The range of the focal operations is denoted by  $P_{rl}$ .

**Definition 1.7.** A *right-strong right Rickart ring* is a right-focal right Rickart ring  $R$  such that

- (a) the right focal operation is normal,
- (b) for all  $p, q \in P_r$ ,

$$pq \in P_r \text{ iff } pq = qp. \quad (1.4)$$

A *right-strong Rickart ring* is a right-focal Rickart ring (i.e., a both-sided Rickart ring equipped with a right focal operation) satisfying the same conditions.

**Definition 1.8.** [Cīrulis, 2016] A *strong Rickart ring* is a left-right-focal Rickart ring such that

- (a) both focal operations are normal and  $a^v = a^\wedge$  and  $a^\wedge = a''$
- (b) for all  $p, q \in P_{rl}$ ,

$$pq \in P_{rl} \text{ iff } pq = qp. \quad (1.5)$$

## Chapter 2

### Partial orders on strong Rickart rings

#### 2.1. The star order

**Definition 2.1.** [Cīrulis, 2016] Let  $R$  be a left-right-focal Rickart ring with normal focal operations such that  $a^\vee = a^\wedge$  and  $a^\wedge = a''$ . We define a relation  $\leq^*$  on  $R$  as follows:  $a \leq^* b$  for  $a, b \in R$  if and only if

$$a^\wedge b = a = ba'' . \quad (2.1)$$

It was proved in [Cīrulis, 2016] that, on a strong Rickart ring, the relation  $\leq^*$  is a partial order. It is called the *star order*.

#### 2.2. The right star orders

**Definition 2.2.** [Marovt et al., 2015, Definitions 10 and 11] Let  $R$  be a unitary  $*$ -ring. We define a binary relation  $*\preceq$  on  $R$  as follows: For  $a, b \in R$ ,  $a *\preceq b$  whenever there exist a projection  $p \in R$  and an idempotent  $e \in R$  such that

- (a) the left annihilator of  $a$  is  $R(1 - p)$ ,
- (b) the right annihilator of  $a$  is  $(1 - e)R$ ,
- (c)  $pa = pb$ ,
- (d)  $ae = be$ .

The relation  $*\preceq$  is called the *strong left star order on  $R$* . Dually, we define the relation  $\preceq^*$  by setting, for  $a, b \in R$ ,  $a \preceq^* b$  if and only if there exist a projection  $q \in R$  and an idempotent  $f \in R$  such that

- (a) the left annihilator of  $a$  is  $R(1 - f)$ ,
- (b) the right annihilator of  $a$  is  $(1 - q)R$ ,
- (c)  $fa = fb$ ,
- (d)  $aq = bq$ . The relation  $\preceq^*$  is called the *strong right star order on  $R$* .

**Definition 2.3.** [Cīrulis, 2015c, Remark 2] Let  $R$  be a right-strong right Rickart ring. The relation  $*\leq$  is defined on  $R$  as follows: For  $a, b \in R$ ,  $a * \leq b$  whenever

$$a = a''b \text{ and } a'' \leq_E b''. \quad (2.2)$$

The relation is called the *weak left star order* on  $R$ . Dually, the *weak right star order* on  $R$  is defined by setting, for  $a, b \in R$ ,  $a \leq_* b$  whenever

$$a = ba'' \text{ and } a'' \leq_E b''. \quad (2.3)$$

### 2.3. Other partial orders

**Definition 2.4.** [Cīrulis, 2017] In a strong Rickart ring, the *diamond order* is the relation  $\overset{\diamond}{\leq}$  defined by  $a \overset{\diamond}{\leq} b$  iff

$$a'' \leq_E b'', a'' \leq_E b'' \text{ and } a = a''ba''. \quad (2.4)$$

**Definition 2.5.** [Rakic, 2015] Let  $R$  be a unitary ring. Let  $\text{ann}_l(a)$  and  $\text{ann}_r(a)$  denote the left and right annihilators of an element  $a$ . Let

$$\mathcal{I}_R := \{x \in R \mid (\exists p \in R)(p^2 = p, \text{ann}_l(x) = \text{ann}_l(p) \text{ and } \text{ann}_r(x) = \text{ann}_r(p))\}. \quad (2.5)$$

For a given element  $x \in R$ , the idempotent  $p$  from Equation (2.5) is unique if it exists. We may therefore denote it by  $p_x$ . Now the relation  $\overset{\#}{\leq}$  is defined on the set  $\mathcal{I}_R$  as follows: For  $a, b \in \mathcal{I}_R$ ,  $a \overset{\#}{\leq} b$  if and only if  $a = bp_a = p_a b$ . The relation  $\overset{\#}{\leq}$  is called the *sharp order*.

## Chapter 3

### Generalizations of the right star order to Rickart rings satisfying a strongness condition

#### 3.1. Introducing the strong right star order on right-strong Rickart rings

**Definition 3.1.** Let  $R$  be a right-strong right Rickart ring and let  $a, b \in R$ . We define a relation  $\preceq^*$  as follows:  $a \preceq^* b$  if and only if there exist an idempotent  $f$  and a closed idempotent  $p$  such that

- ( $\preceq^*$ 1) for all  $x \in R$ ,  $ax = 0$  if and only if  $x \in (1 - p)R$ ,
- ( $\preceq^*$ 2) for all  $x \in R$ ,  $xa = 0$  if and only if  $x \in R(1 - f)$ ,
- ( $\preceq^*$ 3)  $ap = bp$ ,
- ( $\preceq^*$ 4)  $fa = fb$ .

If moreover the ring  $R$  is a both-sided Rickart ring (i.e., a right-strong Rickart ring), then we call the relation  $\preceq^*$  the *strong right star order* on  $R$ .

**Theorem 3.2.** [Cremer, 2024] *On a right-strong Rickart ring, the relation  $\preceq^*$  defined in Definition 3.1 is a partial order.*

We call the relation  $\preceq^*$  the *strong right star order*.

#### 3.2. Right star orders and right space preorders

**Definition 3.3.** (a) Let  $R$  be a right-focal right Rickart ring. The *weak right space preorder*  $\sqsubset_r^w$  is defined as follows: for  $a, b \in R$ ,  $a \sqsubset_r^w b$  if and only if  $a''b'' = a''$ .

(b) Let  $R$  be a ring. The *right space preorder*  $\sqsubset_r$  is defined as follows: for  $a, b \in R$ ,  $a \sqsubset_r b$  if and only if  $Ra \subseteq Rb$ .

- (c) Let  $R$  be a ring. The *strong right space preorder*  $\sqsubset_r^s$  is defined as follows: for  $a, b \in R$ ,  $a \sqsubset_r^s b$  if and only if there exists  $e \in \text{LI}(a)$  such that  $a = eb$  and  $fe = e$  for all  $f \in \text{LI}(b)$ .

**Proposition 3.4.** 1. In a right-strong right Rickart ring,  $a \leq_* b$  if and only if  $a = ba''$  and  $a \sqsubset_r^w b$ .

2. In a right-strong Rickart ring,  $a \preceq_* b$  if and only if  $a = ba''$  and  $a \sqsubset_r^s b$ .

**Theorem 3.5.** Let  $R$  be a right-focal right Rickart ring and let  $\preceq'_*$  be the relation defined on  $R$  as

$$a \preceq'_* b \text{ if and only if } a = ba'' \text{ and } Ra \subseteq Rb. \quad (3.1)$$

If  $R$  is a right-strong right Rickart ring, then  $\preceq'_*$  is a partial order.

**Theorem 3.6.** Let  $a, b$  be elements of a right-strong Rickart ring  $R$ .

- (a) If  $a \preceq_* b$ , then  $a \preceq'_* b$ .  
 (b) If  $a \preceq'_* b$ , then  $a \leq_* b$ .

## Chapter 4

### Minimal upper bounds, joins and meets under suitable conditions

#### 4.1. Joins under the diamond order

**Theorem 4.1.** Let  $R$  be a strong Rickart ring and let  $a, b, x \in R$  such that  $a, b \overset{\diamond}{\leq} x$  and let  $u = (a'' \vee_P b'')x(a'' \vee_P b'')$ . Then the following are equivalent.

- (a)  $a, b \overset{\diamond}{\leq} u$ ,  
 (b)  $u'' = a'' \vee_P b''$  and  $u'' = a'' \vee_P b''$ .  
 (c)  $u$  is a minimal upper bound for  $a$  and  $b$ ,  
 (d)  $u$  is the least upper bound for  $a$  and  $b$  in the initial segment  $[0, x]_{\overset{\diamond}{\leq}}$ .

## 4.2. Conditions for existence of meets and joins under the strong right star order $\preceq^*$

**Lemma 4.2.** [Cremer, 2024, Lemma 5.1] *Let  $R$  be a right-strong Rickart ring and let  $a, b, c \in R$  such that  $a \preceq^* b \preceq^* c$ . If  $a \preceq^* c$ , then also  $a \preceq^* b$ .*

**Theorem 4.3.** [Cremer, 2024, Theorem 5.2] *Let  $R$  be a right-strong Rickart ring and let  $a, b$  be elements of  $R$  such that  $a, b \preceq^* x$  for some  $x \in R$ .*

- (a) *If  $a \wedge^* b \preceq^* x$ , then  $a \wedge^* b$  exists and  $a \wedge^* b = a \wedge^* b$ .*
- (b) *If  $a \wedge^* b$  exists, then it is, under the weak right star order  $\preceq^*$ , the least upper bound in  $[0, x]_{\preceq^*}$ .*

**Theorem 4.4.** [Cremer, 2024, Theorem 5.3] *Let  $R$  be a right-strong Rickart ring and let  $a, b$  be elements of  $R$  such that  $a, b \preceq^* x$  for some  $x \in R$ .*

- (a) *If  $a \vee^* b$  exists, then  $a \vee^* b = a \vee^* b$ .*
- (b) *If  $a \vee^* b \preceq^* x$ , then  $a \vee^* b$  is, under the strong right star order  $\preceq^*$ , the least upper bound of  $a$  and  $b$  in  $[0, x]_{\preceq^*}$ .*

## 4.3. Star-free notions of coherence

**Definition 4.5** (Cīrulis). [Cīrulis and Cremer, 2022, Definition 3.5] *Let  $R$  be a strong Rickart ring. Elements  $a, b \in R$  are said to be weakly coherent if*

$$a \wedge^* b = ab'' \text{ and } b \wedge^* a = ba''. \quad (4.1)$$

**Lemma 4.6.** [Cīrulis and Cremer, 2022]

- (a) *Any pair of closed idempotents is weakly coherent.*
- (b) *Closed idempotents are coherent if and only if they commute.*

**Definition 4.7.** *Let  $R$  be a strong Rickart ring. Elements  $a, b$  of  $R$  are said to be*

- (a) *right Djikić-coherent if  $xa'' = a$  and  $xb'' = b$ ,*
- (b) *left Djikić-coherent if  $a''x = a$  and  $b''x = b$ ,*
- (c) *Djikić-coherent if they are both right Djikić-coherent and left Djikić-coherent.*

## 4.4. Meets and join under the weak right star order for right-coherent elements

**Lemma 4.8.** *Elements  $a, b$  of a right-strong right Rickart ring are right-coherent if and only if they have the meet under the weak right star order and  $a \wedge^* b = ab'' = ba''$ .*

**Theorem 4.9.** *Let  $a, b$  be elements of a right-strong right Rickart ring  $R$  which have an upper bound under the weak right star order  $\leq^*$ . Then the following are equivalent.*

- (a)  $a''b'' = b''a''$ ,
- (b)  $a$  and  $b$  are right-coherent,
- (c)  $ab'' = a \wedge^* b = ba''$ ,
- (d)  $a + ba' = a \vee^* b = b + ab'$ ,
- (e)  $a \vee^* b = a + b - (a \wedge^* b)$ ,
- (f)  $a \vee^* b = ab' + (a \wedge^* b) + ba'$  and  $a''b'' = b''a''$ .

## 4.5. Existence of upper bounds and joins under the star order

**Theorem 4.10.** [C̄irulis and Cremer, 2022] *Let  $R$  be a strong Rickart ring. Elements  $a$  and  $b$  of  $R$  have an upper bound under the star order if and only if they are weakly coherent and are right Djikić-coherent.*

**Theorem 4.11.** [C̄irulis and Cremer, 2022, Theorem 5.4] *The following conditions on elements  $a$  and  $b$  of a strong Rickart ring  $R$  are equivalent:*

- (a)  $a$  and  $b$  are coherent,
- (b)  $a$  and  $b$  have an upper bound under the star order, and  $ab'' \leq^* a, ba'' \leq^* b$ ,
- (c)  $a \vee^* b$  exists, and  $a + ba' = a \vee^* b = b + ab'$ ,
- (d)  $a \vee^* b$  exists, and
  - (1)  $a$  and  $b$  are right coherent,
  - (1)  $a \wedge^* b$  exists and  $a \vee^* b = ab' + a \wedge^* b + ba'$ ,
- (e)  $a \vee^* b$  exists and  $a \vee^* b = a + b - a \wedge^* b$ .

## Chapter 5

### Properties of the strong right star order

#### 5.1. Sectional orthocomplementations for the strong right star order

**Theorem 5.1.** [Cremer, 2024] *A right-strong Rickart ring  $R$  with the strong right star order and sectional orthocomplementations given by*

$$a_x^\perp := x - a = xa' \text{ for } a \preceq^* x, \quad (5.1)$$

*is relatively orthocomplemented. Moreover, if  $a, b \preceq^* x$  and  $a \preceq^* b_x^\perp$  for some  $a, b, x \in R$ , then  $a \vee^* b = a \vee^* b$ .*

#### 5.2. The strong right star orthogonality

**Theorem 5.2.** *Let  $R$  be a right-strong Rickart ring and let  $\perp$  be the orthogonality of the relatively orthocomplemented poset  $\langle R, \preceq^* \rangle$  (where  $\preceq^*$  denotes the strong right star order). Then  $a \perp b$  if and only if*

$$a'' \perp b'' \text{ and } fa = 0 \text{ for some } f \in \text{LI}(b), \quad (5.2)$$

*where  $\perp$  denotes the orthogonality of closed idempotents given by  $p \perp q$  iff  $pq = 0$  iff  $qp = 0$ .*

### 5.3. Isomorphisms for initial segments

#### 5.3.1 Isomorphism onto subset of initial segment of $P_r$

**Theorem 5.3.** *Let  $R$  be a right-strong Rickart ring and let  $x \in R$ . The map*

$$\begin{aligned} \phi_{\leq_*} : [0, x]_{\leq_*} &\rightarrow \{p \in [0, x'']_{\leq} \mid (\exists e \in \text{LI}(xp))ex = xp\} \\ a &\mapsto a'' \end{aligned}$$

*is an order isomorphism which preserves the orthogonality relation  $\perp$ .*

#### 5.3.2 Isomorphism into subset of equivalence classes of $E \times P_r$

**Definition 5.4.** The right preorder on the set of idempotents  $E$  of a right-strong Rickart ring (or a semigroup) is defined by

$$e \lesssim_r f \text{ iff } fe = e. \quad (5.3)$$

**Definition 5.5.** The right equivalence on the set of idempotents of a right-strong Rickart ring (or a semigroup) is the equivalence relation  $\sim_r$  defined by

$$e \sim_r f \text{ iff } (ef = f \text{ and } fe = e) \quad (5.4)$$

For an idempotent  $e$  in a right-strong Rickart ring  $R$ , the equivalence class of  $e$  under the relation  $\sim_r$  is the set  $\text{LI}(e)$ .

Let

$$\tilde{E} := \{\text{LI}(e) \mid e \in E\}. \quad (5.5)$$

The partial order induced on  $\tilde{E}$  by the preorder  $\lesssim_r$  will be denoted by  $\leq$  (i.e., for idempotents  $e$  and  $f$ ,  $\text{LI}(e) \leq \text{LI}(f)$  if and only if  $e \lesssim_r f$ ).

**Theorem 5.6.** *Let  $R$  be a right-strong Rickart ring and let  $x \in R$ . Then the map*

$$\begin{aligned} \phi : [0, x]_{\leq_*} &\rightarrow \tilde{E} \times P_r \\ a &\mapsto (\text{LI}(a), a'') \end{aligned}$$

*is an order embedding and*

$$\text{im } \phi = \{(\text{LI}(e), p) \in \tilde{E} \times P_r \mid \text{LI}(e) \leq \text{LI}(x) \text{ and } p \leq_E x'' \text{ and } ex = xp\}. \quad (5.6)$$

## Chapter 6

### Meet semilattice conditions involving weak BCK-algebras

**Corollary 6.1.** [Cremer and Marovt, ND] *Let  $R$  be a right-strong Rickart ring, and let  $\leq^*$  be either the weak or the strong right star order. Then the following are equivalent.*

- (a) *The poset  $\langle R, \leq^* \rangle$  is a meet semilattice.*
- (b) *The poset  $R$  admits a binary operation  $\searrow_*$  satisfying the conditions*

- ( $\searrow 1$ ) *if  $a \leq b$ , then  $b \searrow_* a = b \ominus a$ .*
- ( $\searrow 2$ )  *$a \ominus (a \searrow_* b)$  is defined and  $a \ominus (a \searrow_* b) \leq b$ ,*
- ( $\searrow 3$ ) *if  $a \leq b$ , then  $c \searrow_* b \leq c \searrow_* a$ .*

- (c) *The poset  $R$  admits a binary operation  $\searrow_*$  such that  $\langle R, \searrow_*, 0 \rangle$  is a weak BCK-algebra and  $b \searrow_* a = b - a$  whenever  $a \leq^* b$ .*

*In the case when these conditions are fulfilled, the binary operations in the latter two items are unique and they coincide, the weak BCK-algebra  $\langle R, \searrow_*, 0 \rangle$  is implicative and the meet of  $a, b \in R$  under the respective version of the right star order is given by  $a \wedge_* b = a - (a \searrow_* b) = (a \searrow_* b)_a^{\perp*}$ .*

**Corollary 6.2.** [Cremer and Marovt, ND] *Let  $R$  be a unitary ring and let  $\mathcal{I}_R$  be the subset of  $R$  as defined in (2.5). Then the following are equivalent.*

- (a) *The poset  $\langle \mathcal{I}_R, \overset{\#}{\leq} \rangle$  is a meet semilattice.*
- (b) *The set  $\mathcal{I}_R$  admits a binary operation  $\searrow_{\#}$  such that  $b \searrow_{\#} a = b - a$  whenever  $a \overset{\#}{\leq} b$  and  $\langle \mathcal{I}_R, \searrow_{\#}, 0 \rangle$  is a weak BCK-algebra.*
- (c) *The set  $\mathcal{I}_R$  admits a binary operation  $\searrow_{\#}$  such that conditions ( $\searrow 1$ ), ( $\searrow 2$ ) and ( $\searrow 3$ ) are satisfied.*

*In the case when these conditions are fulfilled, the binary operations in ?? and ?? are unique and they coincide, the weak BCK-algebra  $\langle \mathcal{I}_R, \searrow_{\#}, 0 \rangle$  is commutative and the meet of elements  $x, y \in \mathcal{I}_R$  is given by  $x \overset{\#}{\wedge} y = x - (x \searrow_{\#} y)$ .*

## Chapter 7

### Special case: Strong semilattice decompositions of reduced Rickart rings

**Example 7.1.** Let  $R$  be a reduced Rickart ring and let  $\langle E, \leq_E \rangle$  be its semilattice of idempotents.

For every idempotent  $e$ , let  $\cdot_e$  be the restriction of the ring multiplication to the so-called  $m$ -domain  $M_e := \{a \in R \mid a^\circ = e\}$ . Let  $\mathcal{M}$  be the family of all the  $m$ -domains  $\langle M_e, \cdot_e \rangle$

For each pair  $e, f \in E$  such that  $e \leq_E f$ , we define a map  $\phi_e^f$  in the following way:

$$\begin{aligned} \phi_e^f : M_f &\rightarrow M_e \\ x &\mapsto xe. \end{aligned} \tag{7.1}$$

Now let  $\Phi$  be the set of all the maps  $\phi_e^f$ .

It is easy to check that  $\langle \mathcal{M}, \Phi \rangle$  is an inverse system of semigroups. We shall denote this inverse system by  $\text{sys } R$ .

The result of the construction described in this Example is very similar to the result of a construction on PP-monoids which can be found in Fountain [1976].

**Corollary 7.2.** [Cremer, 2025]

- (a) Let  $\langle R, +, \cdot, 1 \rangle$  be a reduced Rickart ring. Then  $\langle R, \cdot \rangle = \mathfrak{S}(\text{sys } R)$ .
- (b) Let  $\langle \mathcal{A}, \mathcal{H} \rangle$  be an inverse system of disjoint right-cancellative semigroups over lower semilattice  $S$ . If  $\mathfrak{S}\langle \mathcal{A}, \mathcal{H} \rangle = \langle R, \cdot \rangle$  for some reduced Rickart ring  $R$ , then  $\langle \mathcal{A}, \mathcal{H} \rangle = \text{sys } R$ .

## 7.1. Strong semilattice decomposition using D-semigroups and their inverse systems

### 7.1.1 The inverse system of D-semigroups formed by the $m$ -domains

**Proposition 7.3.** [Cremer, 2025] *Let  $R$  be a reduced Rickart ring, let  $E$  be its semilattice of idempotents and for every pair of idempotents  $e, f$  with  $e \leq_E f$ , let  $\phi_e^f : M_f \rightarrow M_e$  be the map given by Example 7.1 (i.e.,  $\phi_e^f(x) = xe$ ). Let*

$$\mathcal{M}_1^\circ = \{\langle M_e, \cdot_e, \circ_e, e \rangle \mid e \in E\} \quad (7.2)$$

with  $a \cdot_e b := a \cdot b$  and  $a \circ_e := e$ , and let

$$\Phi = \{\phi_e^f \mid e, f \in E \text{ and } e \leq_E f\} \quad (7.3)$$

as in Example 7.1. Then  $\langle \mathcal{M}_1^\circ, \Phi \rangle$  is an inverse system of right-cancellative D-monoids.

**Definition 7.4.** [Cremer, 2025] The inverse system of D-monoids  $\langle \mathcal{M}_1^\circ, \Phi \rangle$  from Proposition 7.3 will be called the *inverse system of D-monoids of the reduced Rickart ring  $R$*  and we will denote it by  $\text{sys}_1^\circ R$ .

### 7.1.2 Strong semilattices of D-monoids

**Definition 7.5.** [Cremer, 2025]

- (a) Let  $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$  be an inverse system of pairwise disjoint D-semigroups over a lower semilattice  $S$ . On the union of all the D-semigroups  $A = \bigcup_{s \in S} A_s$ , we define a binary operation  $\bullet$  as in Equation (B.1) and a unary operation  $\circ$  in the following way: For  $x \in A_s$ ,

$$x^\circ := x_s^\circ. \quad (7.4)$$

Then we write  $A = \mathfrak{S}^\circ \langle \mathcal{A}^\circ, \mathcal{H} \rangle$  and call the algebra  $\langle A, \bullet, \circ \rangle$  a *strong semilattice of D-semigroups* and say that it is *induced by the inverse system  $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$* .

- (b) If  $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$  is even an inverse system of D-monoids over a semilattice  $S$  which has the greatest element  $\top$ , then we can define also a constant  $\mathbf{1}$  by

$$\mathbf{1} := 1_\top. \quad (7.5)$$

We write  $A = \mathfrak{S}_1^\circ [\langle \mathcal{A}^\circ, \mathcal{H} \rangle]$  and call the algebra  $\langle A, \bullet, \circ, \mathbf{1} \rangle$  a *strong semilattice of D-monoids*.

**Proposition 7.6.** [Cremer, 2025]

- (a) Every strong semilattice of  $D$ -semigroups is a  $D$ -semigroup.
- (b) Every strong semilattice of  $D$ -monoids is a  $D$ -monoid.

### 7.1.3 Decomposition of the $D$ -semigroup reduct of an enriched $m$ -domain ring

**Corollary 7.7.** [Cremer, 2025]

- (a) If  $R$  is a reduced Rickart ring equipped with the operation  $\circ$ , then  $\langle R, \cdot, \circ, 1 \rangle = \mathfrak{S}_1^\circ(\text{sys}_1^\circ R)$ .
- (b) Let  $\langle \mathcal{A}^\circ, \mathcal{H} \rangle$  be an inverse system of pairwise disjoint right-cancellative  $D$ -semigroups with identities over some lower semilattice, and let  $\mathcal{A}_1^\circ$  denote the family of  $D$ -monoids obtained from the family of  $D$ -semigroups  $\mathcal{A}^\circ$  by including the identities into the signatures. Suppose  $\mathfrak{S}^\circ \langle \mathcal{A}^\circ, \mathcal{H} \rangle = \langle R, \cdot, \circ, 1 \rangle$  for some enriched reduced Rickart ring  $R$ . Then  $\langle \mathcal{A}_1^\circ, \mathcal{H} \rangle = \text{sys}_1^\circ R$ .

## Conclusion

**Chapter 3** The main result of this chapter is the generalization of the strong right star order to right-strong Rickart rings. This is an improvement from Marovt et al. [2015], where the strong right star order was introduced to Rickart  $*$ -rings. We provide star-free versions of [Marovt et al., 2015, Definition 11, Theorem 9(iii) and Theorem 10].

**Chapter 4** For the diamond order on a strong Rickart ring, we found a sufficient condition for the existence of the join of two elements in an initial segment.

For the strong right star order on a right-strong Rickart ring, we found some conditions concerning the existence of meets and joins in initial segments. These results are weaker than those known for the weak right star order (see [Cīrulis, 2015c, Theorem 4.3, Equations (4.2) and (4.3)]), but they establish conditions under which the meets and joins under both right star orders are equal, which is also interesting.

For the weak right star order on a right-strong right Rickart ring, we found that right precoherence is a sufficient condition for two elements to have the meet, thus improving [Cīrulis, 2015c, Equation (4.2)]. Moreover, we obtain results that help to reveal the intimate connection between the weak right star order and the notions of both right precoherence and right coherence.

For the star order, Theorem 4.5.3 is both an improvement and a star-free version of [Cīrulis, 2015a, Theorem 4.5].

**Chapter 5** One of the main results of this thesis is the description of the order structure of a right-strong Rickart ring  $R$  ordered by the strong right star order  $\preceq^*$ . Namely, we proved that  $\langle R, \preceq^* \rangle$  is a relatively orthocomplemented poset, obtaining a star-free version of previous research by the author which was limited to Rickart  $*$ -rings (see Krēmere [2016]).

All the results from Chapter 5 expand related research on structure of a star ordered strong Rickart ring, as well as a right-strong right Rickart ring equipped with the weak right star order.

**Chapter 6** In this chapter, which contains another main result of the thesis, we obtained more general results which then were applied to some partial orders on rings, including on Rickart rings.

We obtained semilattice conditions for the weak right star order, the strong right star order and the sharp order. These conditions are analogous to the corresponding semilattice condition for the star order given in [Cīrulis, 2015a, Lemma 5.4 and Theorem 5.5]. It is possible that there are other partial orders on rings, Abelian groups or other structures for which Theorem 6.2.1 might be useful – this is an opportunity for further research.

**Chapter 7** In this chapter, for the third main result of this thesis, we study strong semilattice decomposition of certain reducts of reduced Rickart rings.

This builds on a similar result on right PP monoids, see [Fountain, 1976, Theorem 1]. Contrary to that article, we obtained results also in a non-unitary setting (m-domain rings) and we included the additional operations – the unary operation  $\circ$  and the skew meet  $\overleftarrow{\wedge}$  on a reduced Rickart ring – into the strong semilattice construction.

Unfortunately, the results are not obtained for general strong Rickart rings. It could be a direction for further research to extend the results on strong semilattice decompositions to suitable reducts of wider classes of Rickart rings.

# Appendix

# Appendix A

## Posets with certain additional structures

**Definition A.1.** On an orthocomplemented poset, the relation  $\perp$  defined by

$$a \perp b \text{ if and only if } a \leq b^\perp \quad (\text{A.1})$$

(or, equivalently,  $a \perp b$  if and only if  $b \leq a^\perp$ ) is called the *orthogonality relation induced by the orthocomplementation*.

### A.1. Generalizations of orthocomplemented posets

**Definition A.2.** [C̄irulis, 2015c] A poset with a bottom element  $\langle A, \leq, 0 \rangle$  is said to be *sectionally orthocomplemented* if for every  $x \in A$ , there exists an orthocomplementation  $\frac{\perp}{x}$  on the initial segment  $[0, x]_{\leq} = \{a \in A \mid a \leq x\}$ .

**Definition A.3.** [C̄irulis, 2015c] A sectionally orthocomplemented poset  $\langle A, \leq, 0 \rangle$  with sectional complementations denoted by  $\frac{\perp}{x}$  on every initial segment  $[0, x]_{\leq}$  is said to be *relatively orthocomplemented* if for all  $a, b, x, y \in A$ ,

(ro1) if  $a \leq b_{\frac{\perp}{x}}$ , then the join  $a \vee b$  in  $A$  exists (this property is called having *orthojoins*),

(ro2) if  $a \leq x \leq y$ , then  $a_{\frac{\perp}{x}} \leq a_{\frac{\perp}{y}}$ .

### A.2. Weak BCK-algebras

**Definition A.4.** [C̄irulis, 2010] An algebra  $\langle A, \setminus, 0 \rangle$  equipped with a partial order  $\leq$  such that 0 is the bottom element is said to be a *weak BCK-algebra* if, for all  $x, y, z \in A$ ,

(wBCK1)  $x \leq y$  if and only if  $x \setminus y = 0$ ,

(wBCK2) if  $x \setminus y \leq z$ , then  $x \setminus z \leq y$ .

**Definition A.5.** A weak BCK-algebra  $\langle A, \setminus, 0 \rangle$  is said to be

(a) *commutative* (see [Cīrulis, 2013, 2014, 2015b]) if for all  $x, y \in A$ ,

$$x \setminus (x \setminus y) = y \setminus (y \setminus x), \quad (\text{A.2})$$

(b) *implicative* (see [Cīrulis, 2014]) if for all  $x, p, q \in A$ ,

$$\text{if } x \leq p \leq q, \text{ then } p \setminus (q \setminus x) = x. \quad (\text{A.3})$$

## Appendix B

### Semigroup theoretic preliminaries

#### B.1. Inverse systems and strong semilattices

**Definition B.1.** See [Rotman, 2002, page 499]. Let  $\langle S, \leq \rangle$  be a poset and let  $\mathcal{A} = \{A_s | s \in S\}$  be a family of algebras of the same type. Let  $\mathcal{H} = \{h_s^t | s, t \in S \text{ and } s \leq t\}$  be a family of homomorphisms  $h_s^t : A_t \rightarrow A_s$ . Suppose that for all  $r, s, t \in S$

- (a) the homomorphism  $h_t^t$  is the identity map
- (b) if  $r \leq s \leq t$ , then  $h_r^s h_s^t = h_r^t$ .

Then the pair  $\langle \mathcal{A}, \mathcal{H} \rangle$  is called an *inverse system* of the algebras  $A_s$  and the homomorphisms  $h_s^t$  (over the carrier  $S$ ).

**Definition B.2.** [Grillet, 1995, page 75] Let  $\langle S, \wedge \rangle$  be a lower semilattice and let  $\langle \mathcal{A}, \mathcal{H} \rangle$  be an inverse system of pairwise disjoint semigroups over  $\langle S, \wedge \rangle$ .

On the union  $A = \bigcup_{s \in S} A_s$  of all the semigroups, we define an operation  $\bullet$  in the following way. If  $x \in A_s$  and  $y \in A_t$ , and  $\cdot_{s \wedge t}$  denotes the multiplication on the semigroup  $A_{s \wedge t}$ , then

$$x \bullet y := h_{s \wedge t}^s(x) \cdot_{s \wedge t} h_{s \wedge t}^t(y). \quad (\text{B.1})$$

Then we write  $A = \mathfrak{S} \langle \mathcal{A}, \mathcal{H} \rangle$  and call the semigroup  $\langle A, \bullet \rangle$  (the operation  $\bullet$  is known to be associative, see e.g. [Howie, 1995]) a *strong semilattice of semigroups*.

#### B.2. D-semigroups

**Definition B.3.** [Stokes, 2015] A semigroup  $A$  is said to be a *D-semigroup* if there exists some subset  $U$  of its set of idempotents  $E$  such that, for all  $a \in A$ , there exists a smallest  $e \in U$  with the property that  $ea = a$  (smallest in the sense of the standard partial order of idempotents  $\leq_E$  defined by  $e \leq_E f$  iff  $ef = fe = e$ ).

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## List of attended conferences

While conducting the research presented in this thesis, the author has attended six domestic and 11 international conferences. However, the conference talks **C6**, **C7** and **C9** were not directly connected to the topic of this thesis.

- C1** Insa Cremer. Reduced Rickart rings and skew nearlattices. *Algebra and its applications*. Taevaskoja, Estonia, 2017.
- C2** Insa Cremer. Reduced Rickart rings and skew nearlattices. *Topology, Algebra and Categories in Logic 2017*. Prague, Czech Republic, 2017.
- C3** Insa Cremer. Reducēta Rikarta gredzena dekompozīcija pusgrupu stingra pusrežģī (Decomposition of a reduced Rickart ring as a strong semilattice of semigroups). *76th International Scientific Conference of the University of Latvia*. Rīga, Latvia, 2018.
- C4** Insa Cremer. Strong semilattice decomposition of semigroup-bands. AAA96 – 96. *Arbeitstagung Allgemeine Algebra*. Darmstadt, Germany, 2018.
- C5** Insa Cremer. Reducētie Rikarta gredzeni un greizi gandrīzrežģi (Reduced Rickart rings and skew nearlattices). *12th Conference of the Latvian Mathematical Society*. Ventspils, Latvia, 2018.
- C6** Insa Cremer. The variety of reduced Rickart rings. *Summer School on Algebra and Ordered Sets 2018*. Špindlerův Mlýn, Czech Republic, 2018.
- C7** Insa Cremer. Prime ideals in reduced Rickart rings. *International Conference on Algebra and its Applications*. Tartu, Estonia, 2018.
- C8** Insa Cremer. Weak BCK-algebras in strong Rickart rings. AAA98 – 98. *Arbeitstagung Allgemeine Algebra*. Dresden, Germany, 2019.
- C9** Insa Cremer. Reducēto Rikarta gredzenu varietāte (The variety of reduced Rickart rings). *77th International Scientific Conference of the University of Latvia*. Rīga, Latvia, 2019.
- C10** Insa Cremer. Weak BCK-algebra in Rickart rings. *78th International Scientific Conference of the University of Latvia*. Rīga, Latvia, 2020.

- C11** Insa Cremer. Strong semilattice decomposition of  $m$ -domain rings. *AAA100 – 100. Arbeitstagung Allgemeine Algebra*. Online (organized by Jagiellonian University in Kraków), 2021.
- C12** Insa Cremer. Order structure of a strong Rickart ring under the strong right star order. *AAA101 – 101. Arbeitstagung Allgemeine Algebra*. Online (organized by University of Novi Sad), 2021.
- C13** Insa Cremer. Stingro Rikarta gredzenu sakārtojumi un struktūra (Orderings and structure of strong Rickart rings). *79th International Scientific Conference of the University of Latvia*. Online (organized by University of Latvia), 2021.
- C14** Insa Cremer. Stingra pusrežģa dekompozīcija īpašām  $D$ -abundantām  $D$ -pusgrupām (Strong semilattice decomposition for special  $D$ -abundant  $D$ -semigroups). *80th International Scientific Conference of the University of Latvia*. Online (organized by University of Latvia), 2022.
- C15** Insa Cremer. A generalization of differences and its connection to weak BCK-algebras. *AAA103 – 103. Arbeitstagung Allgemeine Algebra*. Tartu, Estonia, 2023.
- C16** Insa Cremer. A generalization of differences and its connection to weak BCK-algebras. *82nd International Scientific Conference of the University of Latvia*. Rīga, Latvia, 2024.
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